Problem Set II: Due Thursday, January 26, 2012

1.) *Electron MHD* (EMHD)

This extended problem introduces you to EMHD and challenges you to apply what you've learned about MHD to understand the structures of a different system of fluid equations. In EMHD, the ions are stationary and the "fluid" is a fluid of electrons. EMHD is useful in problems involving fast Z-pinches, filamentation and magnetic field generation in laser plasmas, Fast Igniter, etc.

The basic equations of EMHD are the electron momentum balance equation

(1) $$\frac{\partial}{\partial t}\underline{\mathrm{v}} + \underline{\mathrm{v}} \cdot \underline{\nabla}\underline{\mathrm{v}} = -\frac{q}{m}\underline{E} - \frac{\nabla P}{\rho} - \frac{q}{mc}\left(\underline{\mathrm{v}} \times \underline{B}\right) - \nu\underline{\mathrm{v}} \,,$$

(2) $$\underline{J} = -nq\underline{\mathrm{v}} \,,$$

and continuity

(3) $$\underline{\nabla} \cdot \underline{J} = 0 \,.$$

Note that here, Ampere's law forces incompressibility of the mass flow $\rho\underline{\mathrm{v}}$. Here $\underline{\mathrm{v}}$ is the electron fluid velocity, ν is the electron-ion collision frequency, $q = |e|$, $m = m_e$. Of course, Maxwell's equations apply, but the displacement current is neglected.

i.) *Freezing-in*

Determine the freezing-in law for EMHD by taking the curl of Eqn. (1) and using the identity

$$\underline{\mathrm{v}} \cdot \underline{\nabla}\underline{\mathrm{v}} = \underline{\mathrm{v}} \times \underline{\omega} - \underline{\nabla}\left(\mathrm{v}^2/2\right).$$

Assume the electrons have $p = p(\rho)$. Approach this problem by trying to derive an equation for "something" which has the structure of the induction equation in MHD. Discuss the physics - what is the "something" and what is it frozen into? In retrospect, why is the frozen-in quantity obvious? How is freezing-in broken?

ii.) *Large Scale Limit*

Show that for $\ell^2 >> c^2/\omega_{pe}^2$, the dynamical equations for EMHD reduce to

$$\frac{\partial B}{\partial t} + \underline{\nabla} \times \left(\frac{\underline{J}}{nq} \times \underline{B} \right) = -\nu \underline{\nabla} \times \left(\frac{\underline{J}}{nq} \right)$$

$$\underline{\nabla} \cdot \underline{J} = 0; \quad \underline{\nabla} \cdot \underline{B} = 0.$$

a.) Show that density remains constant here.

b.) Formulate an energy theorem for EMHD in this limit, by considering the energy content of a "blob" of EMHD fluid.

c.) Discuss the frozen-in law in this limit.

d.) Consider the case of a field $\underline{B} = B(x)\hat{z}$ and $n = n(y)$. Derive a general equation for a field with no tension, and specialize it to the case considered. You may neglect collisions. Prove that (in the general case), magnetic flux is conserved.

e.) Retaining a constant resistivity, solve the resulting equation (from part d.) for B(x) *exactly*, by applying the Hopf-Cole transformation from Burgers' Equation. [N.B.: Whitham, Chapter 4, is a good reference on Burgers' Equation.]

2.) Kulsrud 3.1

3.) Kulsrud 3.2

4.) Kulsrud 3.3

5.) Kulsrud 4.1, paragraph 1

6.) Kulsrud 4.2

7.) Kulsrud 4.4